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Analytical solution of the Schrödinger equation with linear confinement potential

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Abstract. It is shown that the Schrodinger equation with linear confinement potential has an analytical solution The existence of the analytical solution requires additional terms to the potential other than the linear one. The Coulomb-type potential is necessary to get non-zero energy eigenvalues We propose an asymptotically linear potential, and show that we can analytically solve the Schrodinger equation with this potential

1. Introduction

Quarks are believed to be confined to baryons or mesons. Many experimental data show that the confinement potential is proportional to the distance between quarks at large distance. There have been many attempts [1] to deduce confinement potentials directly from the QCD Lagrangian, but no-one has succeeded.

The motion of quarks in a harmonic oscillator type potential [2], proportional to the square of the distance or in the bag [3], has been well studied. On the other hand, hadron masses and heavy quarkonia have been numerically discussed with linear confinement potentials [4-6]. However, an analytical solution of the quark equation of motion with an asymptotically linear potential proportional to the distance between quarks at large distance has not yet been derived

In this paper one analytical solution of the Schrödinger equation with a linear potential proportional to the distance is suggested. It is well known that the solution of the Schrödinger equation with a harmonic oscillator type potential proportional to the square of the distance r^2 can be written as the product of polynomial functions of r and $\exp(-r^2)$, and the one with a Coulomb type potential proportional to the inverse of the distance $1/r$ is written as the product of polynomial functions and $\exp(-r)$. The solution of the Schrödinger equation with a linear confinement potential is naturally supposed to have the form of $\exp(-r^n)$, where $1<n<2$. Actually it is shown that the solution with $n=\frac{3}{2}$ satisfies the Schrödinger equation with an asymptotically linear confinement potential. At the same time, the existence of an analytical solution requires additional terms to the usual confinement potential which is made up of linear+Coulomb terms.

The trial potential we use is the linear+Coulomb type potential. Not only is this potential very popular as a phenomenological potential [4], but also the Coulomb term is necessary for the Schrödinger equation to have a solution with non-zero energy eigenvalues, as discussed later.

2. Deduction of recurrence formula

Spherical symmetric Schrödinger equation for a particle with mass m and orbital angular momentum l is given by

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)-\frac{l(l+1)}{r^2}R+\frac{2m}{\hbar^2}\{E-U(r)\}R=0 \tag{2.1}$$

where $R(r)$ is the radial part of the wavefunction $\psi(r)$;

$$\psi(r)=R(r)Y_l^m(\Omega). \tag{2.2}$$

The energy eigenvalue E of the particle is determined by solving (2 1).

For the spherical symmetric potential $U(r)$, let us consider the confinement potential which is made of a long range linear potential and a short range Coulomb type potential [4]:

$$U(r)=ar-\frac{b}{r} \tag{2.3}$$

Constants a and b are independent of distance r, and their values are determined by experiments so as to reproduce proper physical values of the hadrons. Then equation (2.1) is rewritten as

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=\left\{\frac{2m}{\hbar^2}ar-\frac{2m}{\hbar^2}E-\frac{2m}{\hbar^2}\frac{b}{r}+\frac{l(l+1)}{r^2}\right\}R. \tag{2.4}$$

At large r, equation (2.4) becomes

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2}=\frac{2m}{\hbar^2}aru-\frac{2m}{\hbar^2}Eu. \tag{2 5}$$

where

$$R(r)=\frac{u(r)}{r}. \tag{2.6}$$

If we require the term proportional to r after double differentiation, $u(r)$ must have the form $\exp(-r^{3/2})$ and for the constant term, $u(r)$ must have the form $\exp(-r)$. So the radial part $R(r)$ is assumed to have the form

$$R(r)=\sum_n c_n r^n\exp(-\alpha r^{3/2}-\beta r) \tag{2.7}$$

where α, β and c_n are constants independent of r, and their values are determined to satisfy the Schrödinger equation (2.4). The constant α is of course positive, because the radial wavefunction $R(r)$ must become zero at $r\to\infty$. If the divergence of the radial part $R(r)$ is prohibited at the origin $r=0$, n must be larger than zero.

After substitution of (2 7) into (2.4), the left-hand side of (2.4) becomes

$$\begin{aligned}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)&\\ =\sum_n c_n\{&\tfrac{9}{4}\alpha^2r^{n+1}+3\alpha\beta r^{n+(1/2)}+\beta^2r^n-\tfrac{3}{2}\alpha(2n+\tfrac{5}{2})r^{n-(1/2)}\\ &-2\beta(n+1)r^{n-1}+n(n+1)r^{n-2}\}\exp(-\alpha r^{3/2}-\beta r)\end{aligned} \tag{2.8}$$

and the right-hand side of (2.4) is

$$\sum_n c_n \left\{ \frac{2m}{\hbar^2} ar^{n+1} - \frac{2m}{\hbar^2} Er^n - \frac{2m}{\hbar^2} br^{n-1} + l(l+1)r^{n-2} \right\} \exp(-\alpha r^{3/2} - \beta r). \tag{2.9}$$

Then the equation we have to solve is

$$\sum_n c_n \left[\left(\tfrac{9}{4}\alpha^2 - \frac{2m}{\hbar^2} a \right) r^{n+1} + 3\alpha\beta r^{n+(1/2)} + \left(\beta^2 + \frac{2m}{\hbar^2} E \right) r^n - \tfrac{3}{2}(2n+\tfrac{5}{2})\alpha r^{n-(1/2)} - \left\{ 2(n+1)\beta - \frac{2m}{\hbar^2} b \right\} r^{n-1} + \{ n(n+1) - l(l+1) \} r^{n-2} \right] = 0. \tag{2.10}$$

This equation is rewritten into the following recurrence formula.

$$\{n(n+1) - l(l+1)\}c_n - \left(2n\beta - \frac{2m}{\hbar^2} b \right) c_{n-1} - \tfrac{3}{2}(2n - \tfrac{1}{2})\alpha c_{n-(3/2)} + \left(\beta^2 + \frac{2m}{\hbar^2} E \right) c_{n-2} + 3\alpha\beta c_{n-(5/2)} + \left(\tfrac{9}{4}\alpha^2 - \frac{2m}{\hbar^2} a \right) c_{n-3} = 0. \tag{2.11}$$

3. Solution of the recurrence formula

If the radial wavefunction $R(r)$ is required to be normalizable, c_n must be zero at a finite number of n. If all c_n are zero for $n > k$, the recurrence formula (2.11) gives the following conditions:

$$\tfrac{9}{4}\alpha^2 - \frac{2m}{\hbar^2} a = 0 \tag{3.1}$$

$$3\alpha\beta = 0 \tag{3.2}$$

$$\beta^2 + \frac{2m}{\hbar^2} E = 0 \tag{3.3}$$

$$\tfrac{3}{2}(2k + \tfrac{5}{2})\alpha = 0 \tag{3.4}$$

$$2(k+1)\beta - \frac{2m}{\hbar^2} b = 0 \tag{3.5}$$

and

$$k(k+1) - l(l+1) = 0 \tag{3.6}$$

where it is assumed that $c_k \neq 0$.

Equations (3.1) and (3.5) fix α and β, but these α and β cannot satisfy (3.2) and (3.4). So we introduce additional counter terms in the potential $U(r)$, that is,

$$U(r) = ar - \frac{b}{r} + er^{1/2} + fr^{-1/2} \tag{3.7}$$

where constants e and f are determined to satisfy the Schrödinger equation (2.1). Then (3.2) and (3.4) are rewritten into

$$3\alpha\beta - \frac{2m}{\hbar^2} e = 0 \tag{3.8}$$

and

$$\tfrac{3}{2}(2k+\tfrac{5}{2})\alpha+\frac{2m}{\hbar^2}f=0. \tag{3.9}$$

Now the simultaneous equations we have to solve are (3.1), (3.3), (3.5), (3 6), (3.8) and (3.9). From (3.1) we obtain α;

$$\alpha=\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}a} \tag{3.10}$$

where α is positive. Also, from (3 5) we obtain β·

$$\beta=\frac{m}{(k+1)\hbar^2}b \tag{3.11}$$

It seems as if β was dependent on k. But k is determined by (3.6) as

$$k=l \tag{3 12}$$

Thus

$$\beta=\frac{m}{(l+1)\hbar^2}b \tag{3 13}$$

is constant. The condition (3 12) requires that all c_n should be zero for $n<k$ if we use it in (2.10).

The energy eigenvalue E is given by (3.3);

$$E=-\frac{m}{2(l+1)^2\hbar^2}b^2 \tag{3 14}$$

which depends only on the angular momentum l. Additional constants e and f are determined by (3.8) and (3.9) as

$$e=\frac{b}{l+1}\sqrt{\frac{2m}{\hbar^2}a} \tag{3.15}$$

and

$$f=-(l+\tfrac{5}{4})\sqrt{\frac{2\hbar^2}{m}a}. \tag{3.16}$$

These are dependent on l In order that the Schrödinger equation with the confinement potential given by (2.3) has an analytical solution, the additional terms are necessary, as shown in (3.7), and these terms depend on the orbital angular momentum l.

4. Conclusion

It has been shown that the spherical symmetric Schrödinger equation with the potential

$$U(r)=ar-\frac{b}{r}+\frac{1}{l+1}\sqrt{\frac{2m}{\hbar^2}a}\,br^{1/2}-(l+\tfrac{5}{4})\sqrt{\frac{2\hbar^2}{m}a}\,r^{-1/2} \tag{4.1}$$

has an analytical solution. The radial wavefunction with orbital angular momentum l can be written as

$$R(r)=c_l r^l\exp(-\alpha r^{3/2}-\beta r) \tag{4.2}$$

where

$$\alpha = \frac{2}{3}\sqrt{\frac{2m}{\hbar^2}a} \tag{4.3}$$

and

$$\beta = \frac{m}{(l+1)\hbar^2}b. \tag{4.4}$$

The constant c_l is determined by the normalization condition

The energy eigenvalue E is given by

$$E = -\frac{m}{2(l+1)^2\hbar^2}b^2 \tag{4.5}$$

and depends only on l The state is determined only by the orbital angular momentum l and is degenerate $(2l+1)$-fold.

In the case where the potential does not include the Coulomb term proportional to $1/r$, we may set $b=0$. From (3 5) or (4.4), $\beta=0$ Then, from (4.5), the energy eigenvalue is always zero This potential is the simplest, but is not very interesting. So we propose the asymptotically linear confinement potential (4.1). The most popular phenomenological potential that is used to calculate hadronic properties is the linear+ Coulomb type potential [4]. If necessary, we could add other terms, but the potential (4.1) is the simplest asymptotically linear potential that we can solve analytically. Calculations of hadron properties will be carried out with this potential.

For the purpose of comparison, Eichten's potential [4]

$$U(r) = ar - \frac{b}{r} \tag{4.6}$$

and the new proposed one (4 1) are shown in figure 1 with the same values of the parameters a and b. It will be easy to determine the values of the parameters a and b to reproduce the physical quantities of hadrons.

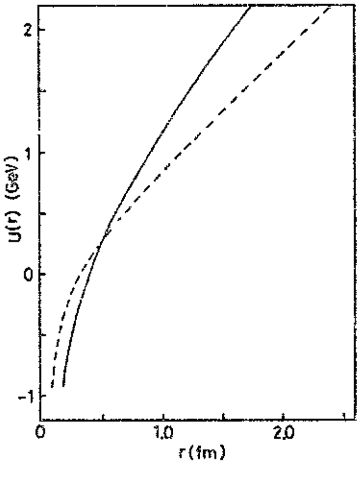

Figure 1. Comparison of the potential (4 1) with $l=0$ (——) and Eichten's potential [4] (4.6) (- - - -) with the same values of $a=0$ 183 GeV^2, $b=0.52$ and mass $m=1.84$ GeV

If the quark state is given by an analytical form, many physical properties of quarks can be easily calculated and understood. It must be worthwhile to investigate quarks in the potential given by (4.1)

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